# §3.5 Connection to Topological Order in Condensed Matter Physics

Topological order is a phase of matter

- which cannot be characterized by a local order parameter (like symmetry breaking)
- where the ground state is degenerate, but its degeneracy cannot be lifted by local perturbations

→ useful for storing quantum information

## 1) Bit-flip code

- Define the so-called "stabilizer Hamiltonian" as follows:

$$H_{Ising} = -J \sum_{k} A_k = -J \sum_{i=1}^{n-1} Z_i Z_{i+1}$$

→ Ising model in 1D with open boundary condition

$\rightarrow$ stabilizer state becomes ground state

- The bit-flip code has stabilizer subspace

$$\{ |00\cdots0\rangle, |11\cdots1\rangle \}$$

$\rightarrow$ ground state is degenerate

bit-flip errors excite ground space to an excited state

- robustness against perturbations:

turn on $\Delta H_{pert} \sim h_x \sum_i X_i$

$\rightarrow$ degeneracy of ground states is not lifted up to $(n-1)$th order of perturbation

$\rightarrow$ $X$ errors of weight up to $n-1$ map code state into an orthogonal subspace $\rightarrow$ groundstate deg. robust!

turn on longitudinal pert. $h_z \sum_i Z_i$

$\rightarrow$ energy between $|00\cdots0\rangle$ and $|11\cdots1\rangle$ is lifted by $nh_z$

→ superposition $\alpha|00\ldots0\rangle + \beta|11\ldots1\rangle$
in groundstate subspace is destroyed

Altogether we see that ground state is only protected if we prohibit longitudinal fields (due to som sym.)

→ "symmetry protected topological order"

- Jordan-Wigner transformation:

define

$$a_{2i-1} = \prod_{k=1}^{i-1} X_k Z_i \,, \quad a_{2i} = \prod_{k=1}^{i-1} X_k Y_i$$

→ $a_k = a_k^\dagger$ (hermitian),

$\{a_k, a_{k'}\} = 2\delta_{k,k'} I$

"Majorana fermions"

→ Ising stabilizer Hamiltonian becomes:

$$H_{Ising} = -J \sum_{j=1}^{n-1} (-i)\, a_{2j}\, a_{2j+1}$$

→ logical operators acting on the groundstates are

$$L_z = a_1 = Z_1 \quad , \quad L_X = a_1 a_{2n} = Y_1 \left( \prod_{k=2}^{n-1} X_k \right) Y_n$$

degree of freedom in degenerate groundstate is unpaired Majorana fermion:

$$|1_L\rangle \equiv a_{2n}|0\rangle + a_{2n} a_1 |0\rangle , \quad |0_L\rangle = a_1|0\rangle + |0\rangle$$

$$\rightarrow L_z |1_L\rangle = -a_{2n} a_1 |0\rangle - a_{2n}|0\rangle = -|1_L\rangle$$

$$L_z |0_L\rangle = |0\rangle + a_1|0\rangle = |0_L\rangle$$

If parity of # fermions is preserved ($a_k a_{k'}$), then there is no symmetry which lifts groundstate degeneracy!

## 2) Kitaev's toric code:

Stabilizer Hamiltonian is given by

$$H_{\text{Kitaev}} = -J \sum_m A_m - J \sum_k B_k$$

→ ground state has 4-fold degeneracy

Errors on code space correspond to excitations

→ two types of excitations:

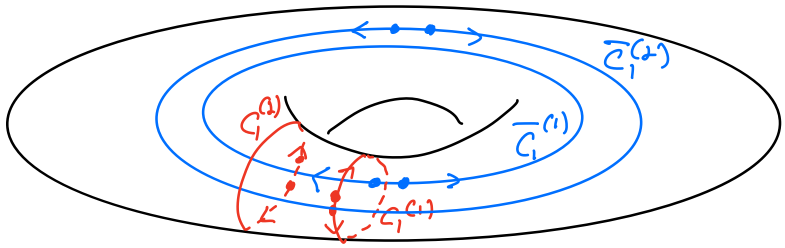
$Z(c_1)$ and $X(\bar{c}_1)$

→ excitations appear at boundaries of error chains $\partial c_1$ and $\partial \bar{c}_1$ (as local energy changes from $-J$ to $+J$ there)

Now consider the following process:

$$(*) \quad Z(c_1^{(2)})\, X(\bar{c}_1^{(2)})\, Z(c_1^{(1)})\, X(\bar{c}_1^{(1)})\, |\psi\rangle = -|\psi\rangle$$

Interpretation:



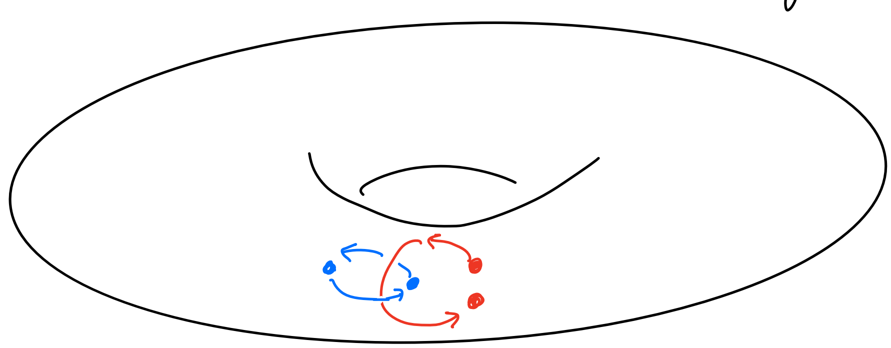

(a) A pair of Z-type excitations is created, moved around torus and annihilated

(b) A pair of X-type excitations is created, moved around torus and annihilated

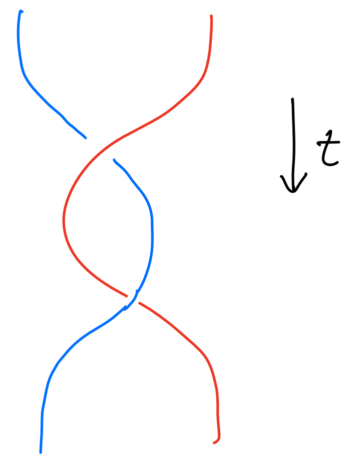
(c) process (a) is repeated

(d) process (b) is repeated

By combining the two red circles to one contractible loop and similarly the two blue circles, one gets:



→ braiding operation of an X-type excitation around a Z-type excitation!



→ after the braiding, a phase factor is applied to the state (*)

→ excitations are neither bosonic nor fermionic
(wave-function would have to stay constant after double exchange)
"anyons" ($\mathbb{Z}_2$ abelian anyons)

More generally, using a finite group $G$, we can define a quantume state $|g\rangle$ $(g \in G)$ in a $|G|$-dim. Hilbert space

→ define 4 types of operators for each $g \in G$:

$$L_+^g = \sum_{h \in G} |gh\rangle\langle h|, \quad L_-^g = \sum_{h \in G} |hg^{-1}\rangle\langle h|,$$

$$T_+^h = |h\rangle\langle h|, \quad T_-^h = |h^{-1}\rangle\langle h^{-1}|$$

Then define the "non-Abelian" toric code: (non-Abelian $G$)

$$H = -J \sum_m A(f_m) - J \sum_k B(v_k)$$

in terms of

$$A(f_m) = \sum_{g_1 g_2 g_3 g_4 = 1} T_-^{g_1}(e^m_{\ell_1}) T_-^{g_2}(e^m_{\ell_2}) T_+^{g_3}(e^m_{\ell_3}) T_+^{g_4}(e^m_{\ell_4})$$

$$B(v_k) = \frac{1}{|G|} \sum_{g \in G} L_+^g(\bar{e}^k_{\ell_1}) L_+^g(\bar{e}^k_{\ell_2}) L_-^g(\bar{e}^k_{\ell_3}) L_-^g(\bar{e}^k_{\ell_4})$$

where the 4 edges $e^m_{\ell_{1,2,3,4}} \in \partial f_m$ and $\bar{e}^k_{\ell_{1,2,3,4}} \in \delta v_k = \partial \bar{f}_k$ are labeled clockwise

→ gives rise to non-Abelian braiding